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1977 J. Phys. A: Math. Gen. 10 1167

(http://iopscience.iop.org/0305-4470/10/7/014)

J. Phys. A: Math. Gen., Vol. 10, No. 7, 1977. Printed in Great Britain. © 1977

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Received 16 February 1977

Abstract. A class of models with spherical constraint is considered. The critical temperatures for a particular case are obtained numerically and are found to depend on a model parameter α contrary to a conjecture of Barrett and Kac. Under certain assumptions, the critical exponents are related to those of an auxiliary model which would have the Ising-like behaviour when $\alpha > 0$ under the universality assumption. Also, the usual spherical model ($\alpha = 0$) results are reproduced from this approach.

1. Introduction and summary

Recently, Barrett and Kac (1975) considered a modified spherical model whose partition function is given by

$$Q_N(\beta, H; \alpha) = \underset{\Sigma x_i^2 = N}{\int \cdots \int} \exp\left(\beta \sum_{i,j} \rho_{ij} x_i x_j + \beta H \sum_i x_i\right) \prod_i (1 + \alpha x_i^2)\, \mathrm{d}\sigma_{\surd N} \tag{1.1}$$

where ρ_{ij} is the interaction matrix, $\beta = 1/kT$, H is the symmetry breaking field, α is a parameter of the model and the integration is taken over the sphere $\Sigma x_i^2 = N$. This model becomes the usual spherical model (Berlin and Kac 1952, Joyce 1972) when $\alpha = 0$ but for $\alpha > 0$, the weighting factor together with the spherical constraint brings the symmetry of the model closer to that of the Ising model.

Barrett and Kac conjectured on the basis of a cluster expansion in powers of α that the model exhibits a phase transition when the corresponding spherical model ($\alpha = 0$) does, and furthermore that the critical temperature $\beta_c(\alpha)$ does not depend on α for at least sufficiently small α. They showed the latter conjecture is indeed true for the mean field case where $\rho_{ij} = J/N$. However, for this mean field interaction, the critical temperatures of the spherical and the Ising model are identical. The fact that $\beta_c(\alpha) = \beta_c(0)$ in this case may, therefore, be fortuitous.

Our purpose here is to investigate the dependence of $\beta_c(\alpha)$ on α for some non-trivial interactions. The one for which one can obtain $\beta_c(\alpha)$, at least numerically, is Dyson's hierarchical interaction (Dyson 1969). With this choice of interaction for ρ_{ij}, we obtain accurate numerical values of critical temperatures as discussed in § 2. Our finding is that they do depend appreciably on α. Although there is still a possibility that $\beta_c(\alpha)$ may be α-independent for a very small range of α, our numerical results strongly indicate that the cluster expansion of the free energy of (1.1) in powers of α does not have a finite radius of convergence at the spherical critical temperature for reasonable interactions.

The other purpose of this paper is to discuss the critical behaviour of the modified spherical model (1.1) under certain assumptions. By introducing an auxiliary system where the spherical constraint of (1.1) is replaced by a Gaussian weighting $\exp(-\beta s\sum_i x_i^2)$ in the integrand, and assuming a thermodynamic scaling form for the auxiliary system, we obtain in § 3 the critical exponents of the modified spherical model from those of the auxiliary system. The latter in turn may be inferred from those of the Ising model if one further assumes that the auxiliary system belongs to the same universality class as in the Ising model (Kadanoff 1976). The effect of the spherical constraint is seen to be exactly the same as that of hidden variables as discussed by Fisher (1968). In other words, the spherical constraint quenches the diverging specific heat of the auxiliary system and makes it finite. Other critical properties are modified accordingly as in Fisher's renormalization of critical exponents (Fisher 1968) and the scaling relations between the exponents are preserved. However, when the auxiliary system does not have a divergent specific heat the critical behaviours of the two systems remain identical.

For the spherical model case $\alpha=0$, some of the arguments of § 3 need to be modified. These modifications, treated separately in § 4, are necessary due to the fact that the renormalization group fixed point of the Gaussian model sits right on a boundary where the thermodynamic limit does not exist. This is also seen as the mechanism which breaks the scaling law in the spherical model for certain cases, e.g. when the dimensionality is greater than 4 for short-range interactions.

2. Critical temperatures

We begin our discussion by introducing the auxiliary partition function

$$\tilde{Q}_N(s,H;a)\equiv\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}\exp\Big(\sum_{i,j}\rho_{ij}x_ix_j-s\sum_i x_i^2+H\sum_i x_i\Big)\prod_i(1+ax_i^2)\,\mathrm{d}x_1\ldots\mathrm{d}x_N \tag{2.1}$$

and free energy

$$\tilde{F}(s,H;a)=\lim_{N\to\infty}N^{-1}\ln\tilde{Q}_N(s,H;a).$$

If we define $\bar{s}(\beta,H;\alpha)$ from the equation

$$\beta=-\frac{\partial}{\partial s}\tilde{F}(s,H\surd\beta;\alpha/\beta)|_{s=\bar{s}} \tag{2.2}$$

then the free energy (including $-\beta$ factors) of the modified spherical model,

$$F(\beta,H;\alpha)=\lim_{N\to\infty}N^{-1}\ln Q_N(\beta,H;\alpha)$$

can be written as (Joyce 1972)

$$F(\beta,H;\alpha)=\beta\bar{s}-\tfrac{1}{2}\ln\beta+\tilde{F}(\bar{s},H\surd\beta;\alpha/\beta). \tag{2.3}$$

We now assume ρ_{ij} is chosen such that a phase transition occurs at $H=0$ and for any α, at least in a finite interval including $\alpha=0$. A phase transition is characterized by the non-analyticity of $\bar{s}(\beta,0;\alpha)$ or equivalently, due to (2.2), by that of $\tilde{F}(s,0;\alpha/\beta)$. If the surface $s^*(\alpha/\beta)$, which we call the critical surface, is the set of points in (s,β,α) space on which $\tilde{F}(s,0;\alpha/\beta)$ is singular, then the line of critical temperatures $\beta_c(\alpha)$ will be determined by the intersection of the critical surface and the surface $\bar{s}(\beta,0;\alpha)$ on which the spherical condition (2.2) is satisfied with $H = 0$. Since s^* depends on α/β only, once $s^*(a)$ is determined by some means for a fixed $\alpha/\beta = a$, $\beta_c(\alpha)$ can then be obtained from (2.2):

$$\beta_c(a) = -\frac{\partial}{\partial s}\tilde{F}(s, 0; a)|_{s=s^*(a)}$$
$$\alpha = a\beta_c(a). \tag{2.4}$$

The critical surface $s^*(a)$ in turn may be determined by the renormalization group theory (Wilson and Kogut 1974); i.e. the value of s which brings the system (2.1) to a fixed point under an appropriate renormalization group transformation.

A model for which the above procedure can actually be carried out exactly is Dyson's hierarchical model (Dyson 1969). Here, the interaction ρ_{ij} is chosen through the identity

$$\sum_{i,j=1}^{N} \rho_{ij}x_ix_j = \sum_{p=1}^{L} 2^{-(1+\sigma)p} \sum_{r=1}^{2L-p} S_{p,r}^2 \tag{2.5}$$

where $N = 2^L$, $S_{p,r} = \Sigma_i x_i$ for $(r-1)2^p + 1 \leqslant i \leqslant 2^p r$ and σ is a parameter which determines the decay of the long-range interaction. The spherical model ($\alpha = 0$; McGuire 1973) and the corresponding Ising model (Dyson 1969) have phase transitions at $H = 0$ for $0 < \sigma < 1$. With this choice of interaction, an exact renormalization of (2.1) can be achieved in various ways (Baker 1972, Bleher and Sinai 1975, Kim and Thompson 1977). For the purpose of the present work, we follow that of Kim and Thompson (1977) where the scaled partition function

$$\tilde{P}_{l+1}(y; s, a) = (\tilde{Q}_n(s, 2^{-(1+\sigma)(l+1)/2+1}y; a))^2 \tag{2.6}$$

$n = 2^l$, $l \geqslant 0$, evolves through the recursion relation

$$\tilde{P}_{l+1}(2^{(1+\sigma)/2}y; s, a) = \left(\pi^{-1/2}\int_{-\infty}^{\infty} \exp[-(x-y)^2]\tilde{P}_l(x; s, a)\,\mathrm{d}x\right)^2 \tag{2.7}$$

under a renormalization process that reduces the spin degree of freedom by one-half. The starting point of the recursion for the system (2.1) is (Kim and Thompson 1977)

$$\tilde{P}_0(y; s, a) = [\pi/(1+s)]^{1/2}[1 + a/2(1+s)][1 + ay^2/(1+s)(1+s+a/2)]$$
$$\times \exp[y^2/(1+s)]. \tag{2.8}$$

Criticality is then defined by the value of $s = s^*(a)$ which brings (2.8) to a fixed point as the renormalization (2.7) is repeated indefinitely. The numerical procedure to obtain $s^*(a)$ is exactly the same as for the Ising model and is explained in Kim and Thompson (1977). The only difference here is that s plays the role of temperature. Furthermore, if $B_0^{(l)}$ is the zeroth-order Hermite coefficient of $\tilde{P}_l$, then it can be shown that

$$\tilde{F}(s, 0; a) = \lim_{l\to\infty} 2^{-l} \ln B_0^{(l)}(s, a) \tag{2.9}$$

so that $\beta_c(a)$ can be determined from the extrapolation of a sequence $\{\beta_l\}$ which is obtained from

$$\beta_l = -\frac{\partial}{\partial s} 2^{-l} \ln B_0^{(l)}(s_l, a) \tag{2.10}$$

where $\{s_l\}$ is the sequence which converges to $s^*(a)$ (see Kim and Thompson 1977).

In tables 1 and 2, we show the values of s^* and β_c thus obtained for various a when $\sigma=\frac{1}{4}$ and $\frac{3}{4}$ respectively. For $\sigma=\frac{1}{4}$ and $a=0$, we could obtain the values of s^* and β_c to ten digit accuracy (see (4.7)). We believe the values for $a>0$ are of the same accuracy even though we have shown in table 1 only the first few significant digits. The deviations of $\beta_c(\alpha)$ from $\beta_c(0)$ are small but are comparable to the difference between $\beta_c(0)$ and that of the Ising model (β_c(Ising) = 0·11602898).

Table 1. Values of β_c and s^* for several values of a when $\sigma=\frac{1}{4}$.

a	s^*	β_c
0	5·2852	0·1166803
1	6·0565	0·1166795
10	8·6713	0·1166239
10^2	12·2987	0·1164618
10^3	13·6761	0·1163981
10^4	13·8675	0·1163895

Table 2. Values of β_c and s^* for several values of a when $\sigma=\frac{3}{4}$.

a	s^*	β_c
0	1·4667	~1·07
10^{-1}	1·5464	~1·052
1	1·8582	0·99240
10	2·3993	0·91566
10^2	2·6606	0·88694
10^3	2·7014	0·88283
Exact values at $a=0$	1·466721	1·07130

The physical fixed point of (2.7) is non-Gaussian when $\sigma>\frac{1}{2}$ and the Gaussian fixed point is thermodynamically unstable in this region. Therefore, when $a=0$ and $\sigma>\frac{1}{2}$, the numerical iteration, being not exactly Gaussian, eventually leads the Gaussian toward the stable non-Gaussian fixed point. Consequently, the above method for $\beta_c(0)$ when $\sigma=\frac{3}{4}$ gives a poor result. However when $a\geqslant 1$, the initial distribution is well away from the Gaussian so that accurate values of β_c are obtained. The dependence of β_c on α for $\sigma=\frac{3}{4}$ is qualitatively the same as for $\sigma=\frac{1}{4}$ if we take note of the fact that β_c(Ising) = 0·77969366 for $\sigma=\frac{3}{4}$.

The fixed points of the $\tilde{P}_l$ transformation are the same as those of the Ising model for all cases considered except for the spherical case $a=0$. Thus for $a>0$, universality is confirmed and consequently, we deduce that the system (2.1) and (2.5) has the same critical behaviour as that of Ising model for any $a>0$, provided that s plays the role of temperature (Kadanoff 1976).

We have also considered the case where the weighting in (1.1) is the sum of two Gaussians centred on ±1 respectively instead of $(1+\alpha x^2)$. Here again, universality is confirmed, and the dependence of the critical temperature on a model parameter, which is a measure of the sharpness of the Gaussian, was found to be qualitatively similar to that of the modified spherical model (1.1).

3. Critical exponents

As the example given in § 2 suggests, the free energy of the auxiliary system (2.1) can be considered to have a phase transition at $H=0$ and $s=s^*(a)$ with s playing the role of temperature. It is natural then to assume that the auxiliary free energy $\tilde{F}(s, H; a)$ for $a>0$ satisfies the usual scaling assumption; that is the 'singular part' $\tilde{F}_s$ of $\tilde{F}$ satisfies the scaling relation

$$\tilde{F}_s(s, H; a)=\lambda^{-d}\tilde{F}_s(s^*(a)+\lambda^{y_1}(s-s^*(a)), \lambda^{y_2}H; a) \tag{3.1}$$

where d is the dimensionality of the system. Here y_1 and y_2 are assumed to be a-independent. If we further assume that the system (2.1) for $a>0$ belongs to the same universality class as the Ising model, then the scaling indices y_1 and y_2 are the same as those of the Ising model. This is most likely true for the hierarchical interaction given by (2.5) since the exact renormalization group takes the system to the same fixed point as in the Ising model for any $a>0$. For other forms of ρ_{ij}, say short-ranged interactions for $d>2$, this assumption is less obvious and to what extent it remains valid is an open question at the present time.

Granted (3.1), we are able to relate the critical behaviour of the modified spherical model (1.1) to that of the auxiliary model (2.1).

Let $\epsilon=s-s^*(a)$. Then from (3.1), $\tilde{F}$ has the form

$$\tilde{F}(\epsilon, 0; a)=A+(y_1/d)B\epsilon^{d/y_1}-C\epsilon+\tfrac{1}{2}D\epsilon^2+\text{higher orders in } \epsilon \tag{3.2}$$

where we have included contributions from the 'regular' part of $\tilde{F}$ which play a crucial role here. Combining (3.2) with (2.2) and (2.4), and assuming $D(a)\neq 0$, we have $C(a)=\beta_c(a)$ and

$$\beta_c(a)-\beta\equiv t=B\epsilon^{d/y_1-1}+D\epsilon+\dots\sim\epsilon^X, \qquad X=\min(1, d/y_1-1). \tag{3.3}$$

We note here that in order for a critical temperature to exist, we require $d/y_1-1\geq 0$.

Using (3.2) and (3.3), we then obtain the following critical behaviour of the modified spherical model. For the specific heat, we have

$$C_H/k=\tfrac{1}{2}+\beta^2(\mathrm{d}\beta/\mathrm{d}s)^{-1}$$
$$\sim\begin{cases}\tfrac{1}{2}-A't^{(2y_1-d)/(d-y_1)} & \text{if } d/y_1<2\\ \tfrac{1}{2}-\beta_c^2/D-A''t^{d/y_1-2}+\text{analytic part in } t & \text{if } d/y_1>2\end{cases} \tag{3.4}$$

where A' and A'' are some constants. Hence, the specific heat remains finite at the critical point and its singular part is described by the exponent α_s given by

$$\alpha_s=2-d/\min(y_1, d-y_1). \tag{3.5}$$

Similarly, for the critical exponents describing the spontaneous magnetization and the susceptibility, we have

$$\beta=(d-y_2)/\min(y_1, d-y_1) \tag{3.6}$$

and

$$\gamma=(2y_2-d)/\min(y_1, d-y_1) \tag{3.7}$$

respectively. Along the critical isotherms, we need to consider the relation between s and H as determined by (2.2). For a finite H, (3.3) is modified to

$$t\sim\epsilon^X-\epsilon^{d/y_1-1}\hbar_+(H/\epsilon^{y_2/y_1}) \tag{3.8}$$

using (3.1) and (3.2). Here $k_+(y)$ is a scaling function which has the property $k_+(0)=0$ and $k_+(y)\sim y^{d/y_2}$ as $y\to\infty$. Hence, at $t=0$, we have

$$H\sim\begin{cases}\epsilon^{y_2/y_1} & \text{if } d/y_1<2\\ \epsilon^{2y_2/d} & \text{if } d/y_1>2\end{cases}\tag{3.9}$$

as $\epsilon\to 0$. Combining this with the scaling form of $\mathrm{d}\tilde{F}/\mathrm{d}H$, we easily obtain

$$\frac{\mathrm{d}}{\mathrm{d}H}F(\beta_c, H; \alpha)\sim H^{(d-y_2)/y_2}$$

irrespective of y_1 so that

$$\delta=y_2/(d-y_2).\tag{3.10}$$

The critical exponents obtained above satisfy the scaling laws, $\alpha_s+2\beta+\gamma=2$ etc.

Our results for the modified spherical model may be summarized by expressing the scaling assumption in the form

$$F_s(\beta, H; \alpha)=\lambda^{-d}F_s(\beta_c+\lambda^{y_1'}(\beta-\beta_c(\alpha)), \lambda^{y_2}H; \alpha)\tag{3.11}$$

for the singular part of $F(\beta, H; \alpha)$ where

$$y_1'=\min(y_1, d-y_1).\tag{3.12}$$

It is then easily seen that the effect of the spherical condition (2.2) is to change the scaling index y_1 to $d-y_1$ whenever $2y_1>d$, or equivalently, whenever the specific heat of the auxiliary system diverges. This is exactly the same as Fisher's renormalization of critical exponents by hidden variables (Fisher 1968). Furthermore, this is also true for spin–spin correlations (Kac and Thompson 1977) and the correlation length exponent becomes, due to (3.3)

$$\nu=1/y_1'=\max[1/y_1, 1/(d-y_1)]\tag{3.13}$$

if one assumes hyperscaling. However, the exponent η remains the same as in the auxiliary system (Kac and Thompson 1977) which is given by

$$\eta=2+d-2y_2\tag{3.14}$$

under hyperscaling. These results are in fact not restricted to the particular weighting $(1+\alpha x^2)$ in (1.1) and should apply equally well to any reasonable weightings such as that mentioned at the end of § 2.

If we consider, as an example, the one-dimensional system whose interaction is given by (2.5), then, as mentioned before, the scaling indices y_1 and y_2 are the same as those of the Ising model. From the numerical values of y_1 (Bleher and Sinai 1975, Baker and Golner 1977, Kim and Thompson 1977), we find that $y_1>1-y_1$ for the range of σ, $\frac{1}{2}<\sigma<\sigma_0$ where $\sigma_0\approx 0{\cdot}72$ is determined numerically. Therefore, for this range of σ, the critical behaviour of the modified spherical model, from (3.12), is different from that of the Ising model. We plot in figure 1 the resulting values of y_1' for the range of $\frac{1}{2}<\sigma<1$. This together with

$$y_2=(1+\sigma)/2\tag{3.15}$$

determines all critical exponents as discussed above for $\frac{1}{2}<\sigma<1$, including ν and η, since hyperscaling also holds for this range of σ.

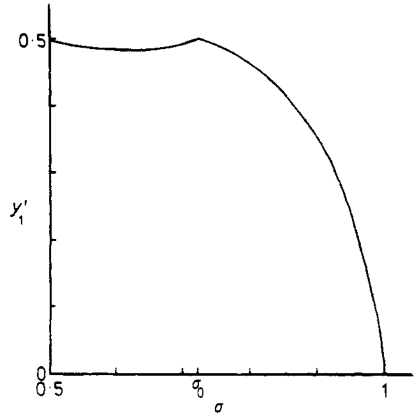

Figure 1. The scaling index y_1' of the modified spherical model with the interaction (2.5) against σ, the long-range interaction parameter.

4. Spherical model

In this section, we apply the discussion given in §§ 2 and 3 to the pure spherical model where the exact solution is known. However, in this case we need an *ad hoc* consideration because the scaling form (3.1) for the Gaussian model does not apply when $s < s^*(a)$ or $s = s^*(a)$ and $H > 0$. As an illustration, we again consider the long-range interaction (2.5). For this interaction, the partial trace over half of the spin degrees of freedom in (2.1) yields the relation (Kim and Thompson 1977)

$$\tilde{Q}_N(s, H; 0) = (2^\sigma \pi/s)^{N/4} \tilde{Q}_{N/2}(2^\sigma s - 1, 2^{(1+\sigma)/2} H; 0) \tag{4.1}$$

so that

$$\tilde{F}(s, H; 0) = \tfrac{1}{2}\tilde{F}(2^\sigma s - 1, 2^{(1+\sigma)/2} H; 0) + \tfrac{1}{4} \ln (2^\sigma \pi/s) \tag{4.2}$$

as long as the thermodynamic limit exists. However, the thermodynamic limit does not exist in this case when $s < (2^\sigma - 1)^{-1}$ or $s = (2^\sigma - 1)^{-1}$ and $H > 0$. Therefore the scaling form (3.1) holds with $d = 1$, $y_1' = \sigma$, $y_2 = (1+\sigma)/2$ and $s^*(0) = (2^\sigma - 1)^{-1}$ and has meaning only when $s > s^*(0)$. This is also true for the d-dimensional short-range interactions in which case we have $y_1 = 2$, $y_2 = 1 + d/2$ and $s^*(0)$ is the limit of the largest eigenvalues of ρ_{ij} (Wilson and Kogut 1974).

Consequently the considerations given in § 3 for the spontaneous magnetization and the magnetization along the critical isotherm for $H > 0$ cannot be applied for the spherical model. To remedy this situation, we make use of the known H-dependence of the free energy of the Gaussian models; that is, we use

$$\tilde{F}(s, H; 0) - \tilde{F}(s, 0; 0) \sim H^2/\epsilon, \qquad \epsilon > 0, \tag{4.3}$$

where $\epsilon = s - s^*(0)$ as before. Then (3.8) is changed to

$$t \sim \epsilon^X - AH^2/\epsilon^2 \tag{4.4}$$

where A is some constant and the magnetization becomes

$$m = \frac{\mathrm{d}}{\mathrm{d}H} F(\beta, H; 0) \sim \frac{H}{\epsilon}. \tag{4.5}$$

From (4.4) and (4.5), we then find $-t \sim m^2$ as $H \to 0$ so that the exponent $\beta = \frac{1}{2}$ irrespective of X. Also at $t = 0$, we have

$$m \sim H^{X/(X+2)}$$

hence,

$$\delta = 1 + 2/X = \max[3, (d + y_1)/(d - y_1)]. \tag{4.6}$$

The rest of the discussion given in § 3 including (3.13) and (3.14) remains valid for this case because only the 'high-temperature' side of the scaling form (3.1) is utilized. Finally if we substitute the values of y_1 and y_2 given above into these formulae, we get complete agreement with the exact solutions (Joyce 1972, McGuire 1973).

Lastly, we note here that we can obtain the critical temperature of the spherical model using only the renormalization group relation (4.2) without actually solving it for the detailed expression of $\tilde{F}$. To see this, we take the derivative of (4.2) with respect to s at $H = 0$ and use (2.4) to obtain

$$\beta_c(0) = 2^{\sigma-1}\beta_c(0) + (4s^*(0))^{-1}.$$

Then, using $s^*(0) = (2^\sigma - 1)^{-1}$ which is also obtained from (4.2), we immediately obtain

$$\beta_c(0) = (2^\sigma - 1)/4(1 - 2^{\sigma-1}) \tag{4.7}$$

as in McGuire (1973).

Acknowledgment

We are grateful to the Australian Research Grants Commission for its support.

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